Tunable Flux through a Synthetic Hall Tube of Neutral Fermions

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Hall tube with a tunable flux is an important geometry for studying quantum Hall physics, but its experimental realization in real space is still challenging. Here, we propose to realize a synthetic Hall tube with tunable flux in a one-dimensional optical lattice with the synthetic ring dimension defined by atomic hyperfine states. We investigate the effects of the flux on the system topology and study its quench dynamics. Utilizing the tunable flux, we show how to realize topological charge pumping. Finally, we show that the recently observed quench dynamics in a synthetic Hall tube can be explained by the random flux existing in the experiment.

Introduction.—Ultracold atoms are emerging as a promising platform for the study of condensed matter physics in a clean and controllable environment [1, 2]. The capability of generating artificial gauge fields and spin-orbit coupling using light-matter interaction [3–18] offers new opportunity for exploring topologically nontrivial states of matter [19–25]. One recent notable achievement was the realization of Harper-Hofstadter Hamiltonian, an essential model for quantum Hall physics, using laser-assisted tunneling for generating artificial magnetic fields in two-dimensional (2D) optical lattices [26–28]. Moreover, synthetic lattice dimension defined by atomic internal states [29–37] provides a new powerful tool for engineering new high-dimensional quantum states of matter with versatile boundary manipulation [32, 33].

Nontrivial lattice geometries with periodic boundaries (such as a torus or tube) allow the study of many interesting physics such as the Hofstadter's butterfly [38] and Thouless pump [39–44], where the flux through the torus or tube is crucially important. In a recent experiment [37], a synthetic Hall tube has been realized in a 1D optical lattice and interesting quench dynamics have been observed, where the flux effect was not considered. More importantly, the flux through the tube, determined by the relative phase between Raman lasers, is spatially non-uniform and random for different iterations of the experiment, yielding major deviation from the theoretical prediction. The physical significance and experimental progress raise two natural questions: can the flux in the synthetic Hall tube be controlled and tuned? If so, can such controllability lead to the observation of nontrivial topological phases and dynamics?

In this Letter, we address these important questions by proposing a simple scheme to realize a controllable flux Φ through a three-leg synthetic Hall tube and studying its quench dynamics and topological pumping. Our main results are:

i) We use three hyperfine ground spin states, each of which is dressed by one far-detuned Raman laser, to realize the synthetic ring dimension of the tube. The flux Φ can be controlled simply by varying the polarizations of the Raman lasers [11]. The scheme can be applied to both Alkali (e.g., potassium) [11–15] and Alkaline-earth(-like) atoms (e.g., strontium, ytterbium) [45].

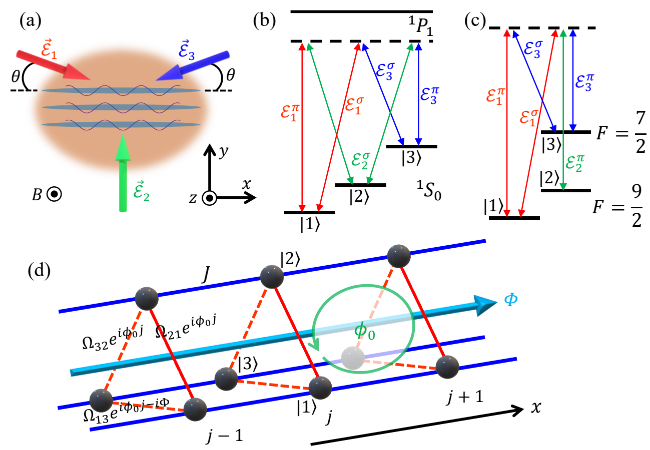

FIG. 1: (a) Schematic of the experimental setup for tunable flux through the synthetic Hall tube. Three Raman lasers $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ generate the couplings along the synthetic dimension spanned by atomic hyperfine states. (b,c) Three hyperfine ground states and the corresponding two-photon Raman transitions for alkaline-earth(-like) atoms (b) and for alkali atoms (c). (d) Synthetic Hall tube with a uniform flux ϕ_0 on each side plaquette and Φ through the tube.

ii) The three-leg Hall tube is characterized as a 2D topological insulator with Φ playing the role of the momentum along the synthetic dimension. The system reduces to a 1D topological insulator at $\Phi = 0$ and π, where the winding number is quantized and protected by a generalized inversion symmetry.

iii) The tunable Φ allows the experimental observation of topological charge pumping in the tube geometry.

iv) We study the quench dynamics with a tunable flux and show that the experimental observed quench dynamics in [37] can be better understood using a random flux existing in the experiment.

The model.—We consider an experimental setup with cold atoms trapped in 1D optical lattices along the x-direction, where transverse dynamics are suppressed by deep optical lattices, as shown in Fig. 1a. The bias magnetic field is along the z direction to define the quantization axis. Three far-detuned Raman laser fields $\vec{\mathcal{E}}_s$, propagating in the x-y plane, are used to couple three atomic hyperfine ground spin states, with each state dressed by one Raman laser, as shown in Figs. 1b and 1c for alkaline-earth(-like) (e.g., strontium, ytterbium) and alkali (e.g., potassium) atoms, respectively. The three spin states form three legs of the synthetic tube system as shown in Fig. 1d, and the tight-binding Hamiltonian is written as

$$H = \sum_{j;s\neq s'} \widetilde{\Omega}_{ss';j} c^{\dagger}_{j,s} c_{j,s'} - \sum_{j;s} (J c^{\dagger}_{j,s} c_{j+1,s} + H.c.), \quad (1)$$

where $\widetilde{\Omega}_{ss';j} = \Omega_{ss'} e^{i\phi_{j;ss'}}$, $c^{\dagger}_{j,s}$ is the creation operator with j, s the site and spin index. J and $\Omega_{ss'}$ are the tunneling rate and Raman coupling strength, respectively. For alkaline-earth(-like) atoms, we use three states in the ${}^1S_0$ ground manifold to define the synthetic dimension. The long lifetime ${}^3P_0$ or ${}^3P_1$ levels are used as the intermediate states for the Raman process (see Fig. 1b) such that $\delta m_F = \pm 2$ Raman process does not suffer the heating issues [36, 37]. While for alkaline atoms (see Fig. 1c), we choose three hyperfine spin states $|F, m_F\rangle$, $|F, m_F - 1\rangle$ and $|F-1, m_F\rangle$ from the ground-state manifold to avoid $\delta m_F = \pm 2$ Raman process, so that far-detuned Raman lasers can be used to reduce the heating [11, 12].

The laser configuration in Fig. 1a generates a uniform magnetic flux penetrating each side plaquette as well as a tunable flux through the tube. Each Raman laser may contain both z-polarization (responsible for π transition) and in-plane-polarization (responsible for σ transition) components, which can be written as $\vec{\mathcal{E}}_s = \hat{\mathbf{e}}_\pi \mathcal{E}^\pi_s + \hat{\mathbf{e}}_\sigma \mathcal{E}^\sigma_s$. For alkaline-earth(-like) atoms, we choose $\mathcal{E}^\pi_2 = 0$ so that $\widetilde{\Omega}_{12;j} \propto \mathcal{E}^\sigma_2 \mathcal{E}^{\pi *}_1$, $\widetilde{\Omega}_{23;j} \propto \mathcal{E}^\pi_3 \mathcal{E}^{\sigma *}_2$ and $\widetilde{\Omega}_{31;j} \propto \mathcal{E}^\sigma_1 \mathcal{E}^{\sigma *}_3$. The corresponding Raman coupling phases are $\phi_{j;21} = j\phi_0 + \varphi^\pi_1 - \varphi^\sigma_2$, where $\varphi^{\pi,\sigma}_s$ are the (π, σ)-component phases of the s-th Raman laser at site $j = 0$, and $\phi_0 = k_R d_x \cos(\theta)$ gives rise to the magnetic flux penetrating the side plaquette of the tube, with k_R the recoil momentum of the Raman lasers and d_x the lattice constant. Similarly, we have $\phi_{j;32} = j\phi_0 + \varphi^\sigma_2 - \varphi^\pi_3$, $\phi_{j;13} = -2j\phi_0 + \varphi^\sigma_3 - \varphi^\sigma_1$. To obtain a uniform magnetic flux for each side plaquette of the tube, we choose the incident angle θ such that $\phi_0 = 2\pi/3$. The phases $\varphi^{\sigma,\pi}_s$ determine the flux through the tube $\Phi = \Delta\varphi_3 - \Delta\varphi_1$, where $\Delta\varphi_s = \varphi^\pi_s - \varphi^\sigma_s$ is the phase difference between two polarization components of the s-th Raman laser. The phase differences $\Delta\varphi_s$ can be simply controlled and tuned using wave plates, and they do not depend on the transverse positions y and z.

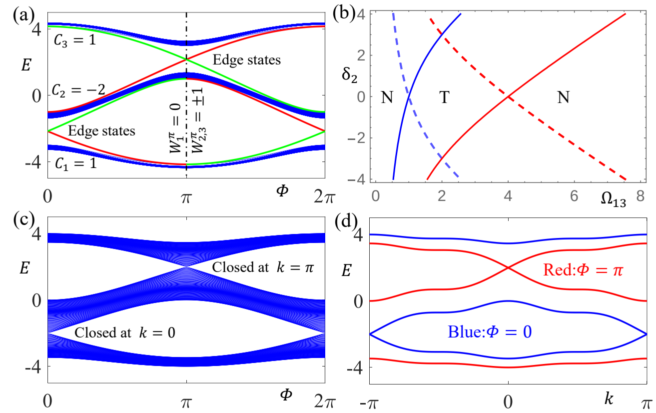

FIG. 2: (a) Band structures in the topological phase. (b) Phase diagram in the Ω_{13}-δ_2 plane, with solid (dashed) lines the boundary between topological (T) and normal (N) phases for the upper (lower) gap. (c) and (d) Band structures at the phase boundary (blue lines) with $\delta_2 = 0$ and $\Omega_{13} = 1$. Common parameters: $\Omega = 2$ with energy unit J.

For alkali atoms, we choose $\mathcal{E}^\sigma_2 = 0$, yielding $\widetilde{\Omega}_{12;j} = \alpha_{12} \mathcal{E}^\pi_2 \mathcal{E}^{\sigma *}_1$, $\widetilde{\Omega}_{23;j} = \alpha_{23} \mathcal{E}^\sigma_3 \mathcal{E}^{\pi *}_2$ and $\widetilde{\Omega}_{31;j} = \beta_{31} \mathcal{E}^\sigma_1 \mathcal{E}^{\pi *}_3 + \alpha_{31} \mathcal{E}^\pi_1 \mathcal{E}^{\sigma *}_3$, with $\alpha_{s,s'}, \beta_{s,s'}$ determined by the transition dipole matrix. We further consider $(\mathcal{E}^\pi_3, \mathcal{E}^\sigma_1) \ll (\mathcal{E}^\sigma_3, \mathcal{E}^\pi_1) \ll \mathcal{E}^\pi_2$, thus $\widetilde{\Omega}_{31;j} \simeq \alpha_{31} \mathcal{E}^\pi_1 \mathcal{E}^{\sigma *}_3$ with amplitudes $\Omega_{21} \sim \Omega_{32} \sim \Omega_{13}$. Similar as the alkaline-earth(-like) atoms, we obtain uniform magnetic flux $\phi_0 = 2\pi/3$ for tube side by choosing $k_R d_x \cos(\theta) = 2\pi/3$. The flux through the tube becomes $\Phi = \Delta\varphi_3 + \Delta\varphi_1$, which can also be tuned at will through the polarization control. With proper gauge choice, we can set the tunneling phases as $\phi_{j;21} = \phi_{j;32} = j\phi_0$ and $\phi_{j;13} = j\phi_0 + \Phi$, as shown in Fig. 1d.

Phase diagram.— The Bloch Hamiltonian in the basis $[c_{k,1}, c_{k,2}, c_{k,3}]^T$ reads

$$H_k = \begin{bmatrix} -2J\cos(k - \phi_0) & \Omega_{21} & \Omega_{13} e^{-i\Phi} \\ \Omega_{21} & -2J\cos(k) & \Omega_{32} \\ \Omega_{13} e^{i\Phi} & \Omega_{32} & -2J\cos(k + \phi_0) \end{bmatrix}, \quad (2)$$

with k the momentum along the real-space lattice. For $\Omega_{21} = \Omega_{32} = \Omega_{13}$, the above Hamiltonian is nothing but the Harper-Hofstadter Hamiltonian with Φ the effective momentum along the synthetic dimension, and $\phi_0 = 2\pi/3$ is the flux per plaquette. The topology is characterized by the Chern number [46]

$$C_n = \frac{i}{2\pi} \int dk d\Phi \langle \partial_\Phi u_n | \partial_k | u_n \rangle - \langle \partial_k u_n | \partial_\Phi | u_n \rangle, \quad (3)$$

where $|u_n\rangle$ is the Bloch states of the n-th band, satisfying $H_k(\Phi)|u_n(\Phi, k)\rangle = E_n(\Phi, k)|u_n(\Phi, k)\rangle$. In Fig. 2a, we plot the band structures as a function of Φ with an open boundary condition along the real-lattice direction. There are three bands, and two gapless edge states (one

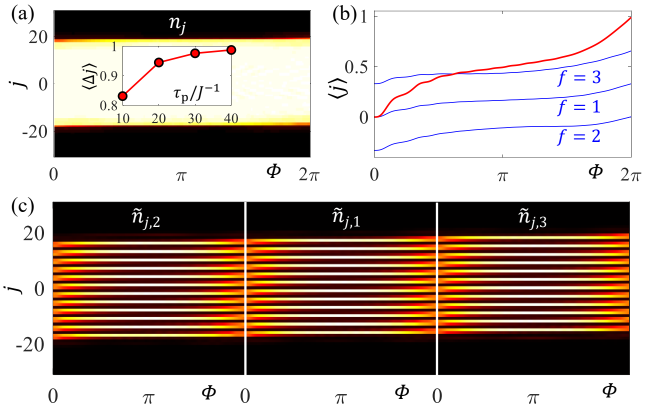

FIG. 3: (a) Total density distribution during one pump cycle. Inset shows the non-adiabatic effects on the center-of-mass shift. (b) Center-of-mass (red line) and the rotated-spin contributions (blue lines) during one pump cycle. The blue lines are $\sum_j j\widetilde{n}_{j,f}(t)/N$ with $f = 1,2,3$ as labeled. (c) Rotated spin-density distributions during one pump cycle.

at each ends) in each gap. The two edge states cross only at $\Phi = 0$ and $\Phi = \pi$, where the tube belongs to a 1D topological insulator with quantized winding number (Zak phase) [47]

$$W_n^{0,\pi} = \frac{1}{\pi}\oint dk \langle u_n|\partial_k|u_n\rangle\big|_{\Phi=0,\pi}. \tag{4}$$

The winding number is protected by a generalized inversion symmetry $\mathcal{I}H_k\mathcal{I}^{-1} = H_{-k}$, where the inversion symmetry $\mathcal{I}$ swaps spin states $|1\rangle$ and $|3\rangle$ [48].

The Chern number and winding number are still well defined even when the Raman couplings have detunings and/or the coupling strength $\Omega_{ss'}$ are nonequal. The changes in these Raman coupling parameters drive the phase transition from topological to trivial insulators. The detuning can be introduced by including additional terms $\sum_{j;s}\delta_s c^\dagger_{j,s}c_{j,s}$ in the Hamiltonian Eq. 1. Hereafter we will fix $\Omega_{21} = \Omega_{32} \equiv \Omega$ and $\delta_1 = \delta_3 = 0$ for simplicity. The phase diagram in the Ω_{13}-δ_2 plane is shown in Fig. 2b. The solid (dashed) lines are the phase boundaries corresponding to the gap closing between two lower (upper) bands, with topological phases between two boundaries. At the phase boundaries, the corresponding band gaps close at $\Phi = 0$ ($\Phi = \pi$) for the two lower (upper) bands, as shown in Fig. 2c. In addition, for the two lower (upper) bands, the gap closes at $k = 0$ and $k = \pi$ ($k = \pi$ and $k = 0$) on the right and left boundaries, respectively, as shown in Fig. 2d. The gaps reopen in the trivial phase with the disappearance of edge states.

Topological pumping.— The three-leg Hall tube is a minimal Laughlin's cylinder. When the flux through the tube is adiabatically changed by 2π, the shift of Wannier-function center is proportional to the Chern number of the corresponding band [39–42]. Therefore all particles are pumped by C site (with C the total Chern number of the occupied bands) as Φ changes by 2π, i.e., C particles are pumped from one edge to another. Given the ability of controlling the flux through the tube, we can measure the topological Chern number based on topological pumping by tuning the flux Φ adiabatically (compared to the band gaps).

Here we consider the Fermi energy in the first gap with only the lowest $C = 1$ band occupied, and study the zero temperature pumping process (the pumped particle is still well quantized for low temperature comparing to the band gap) [49]. The topological pumping effect can be identified as the quantized center-of-mass shift of the atom cloud [49–52] in a weak harmonic trap $V_{\text{trap}} = \frac{1}{2}v_{\text{T}}\, j^2$. The harmonic trap strength $v_{\text{T}} = 0.008J$ and the atom number $N = 36$ are chosen such that the atom cloud has a large insulating region (corresponding to one atom per unit-cell) at the trap center. For simplicity, we set $\Omega_{ss'} = J$ and $\delta_s = 0$ for all s, s', choose the gauge as $\phi_{j;21} = \phi_{j;32} = j\phi_0$, $\phi_{j;13} = j\phi_0 - \Phi$, and change Φ slowly (compared to the band gap) as $\Phi(t) = \frac{2\pi t}{\tau_{\text{p}}}$. In Figs. 3a and 3b, we plot the total density distribution $n_j(t)$ and the center-of-mass $\langle j(t)\rangle \equiv \sum_j jn_j(t)/N$ shift during one pumping circle with $\tau_{\text{p}} = 40J^{-1}$, and we clearly see the quantization of the pumped atom $\langle\Delta j\rangle \equiv \langle j(\tau_{\text{p}})\rangle - \langle j(0)\rangle = 1$. The atom cloud shifts as a whole with $n_j(t) = 1$ near the trap center. The inset in Fig. 3a shows non-adiabatic effect (finite pumping duration τ_{p}) on the pumped atom.

The atoms are equally distributed on the three spin states [i.e., $n_{j,s}(t) \equiv \langle c^\dagger_{j,s}c_{j,s}\rangle = \frac{1}{3}n_j(t)$]. To see the pumping process more clearly, we can examine the spin densities in the rotated basis $\widetilde{n}_{j,f}(t) \equiv \langle\widetilde{c}^\dagger_{j,f}\widetilde{c}_{j,f}\rangle$, with $\widetilde{c}_{j,f} = \frac{1}{\sqrt{3}}\sum_s c_{j,s}e^{is\frac{2f\pi-\Phi}{3}}$. The Hamiltonian in these basis reads $H = \sum_{f=1}^{3} H_f$, where

$$H_f = 2J\cos(K_{j,f,\Phi})\widetilde{c}^\dagger_{j,f}\widetilde{c}_{j,f} + (J\widetilde{c}^\dagger_{j,f}\widetilde{c}_{j,f+1} + H.c.) \tag{5}$$

is the typical Aubry-André-Harper (AAH) Hamiltonian [53, 54] with $K_{j,f,\Phi} = \frac{2\pi(f+j)-\Phi}{3}$. The bulk-atom flow during the pumping can be clearly seen from the spin densities $\widetilde{n}_{j,f}(t)$, as shown in Fig. 3c. Each spin component contributes exactly one third of the quantized center-of-mass shift (see Fig. 3b). The quantized pumping can also be understood by noticing that the AAH Hamiltonians H_s are permutated as $H_1 \to H_3 \to H_2 \to H_1$ after one pump circle. Each H_f returns to itself after three pump circles with particles pumped by three sites (since the lattice period of H_f is 3). Therefore, for the total Hamiltonian H, particles are pumped by one site after one pump circle. The physics for different values of $\Omega_{ss'}$ and δ_s are similar, except that the rotated basis $\widetilde{c}_{j,f}$ may take different forms.

In the presence of strong interaction, which is long-

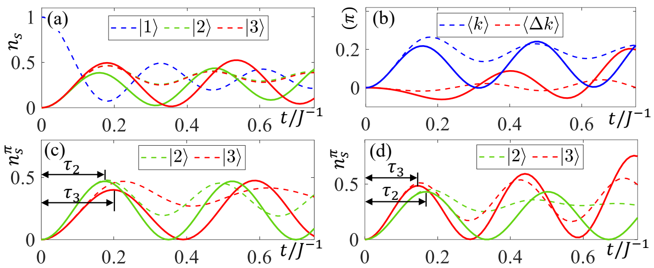

FIG. 4: Quench dynamics for $\Phi = 0$ (solid lines) and averaged over random Φ (dashed lines). Time evolution of spin populations (a) and averaged momenta (b) with $\Omega_{13} = \Omega$. Time evolution of spin populations at $k = \pi$ with $\Omega_{13} = 5$ in (c) and $\Omega_{13} = 7.5$ in (d). $n_s^\pi = \frac{n_s(\pi)}{\sum_{s'} n_{s'}(\pi)}$, τ_2 and τ_3 cross at $\Omega_{13}^c = 5.8$. Both gaps are topological in (a) and (b), and only the upper (lower) gap is topological in (c) [(d)]. Common parameters: $\Omega = 6.15$, $\delta_2 = -0.2\Omega$ with energy unit J.

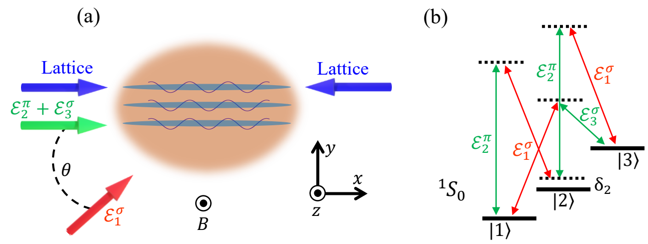

FIG. 5: (a) Schematic of the experimental setup in [37]. (b) The three hyperfine ground states and the corresponding two-photon Raman transitions.

range in the synthetic dimension, our scheme offers an ideal platform for studying exotic fractional quantum Hall phases and topological fractional charge pumping [43].

Quench dynamics.—Besides topological pumping, the quench dynamics of the system can also be used to demonstrate the presence of gauge field ϕ_0 and detect the phase transitions [37]. Here we study how Φ affects the quench dynamics by considering that all atoms are initially prepared in state $|1\rangle$, then the inter leg couplings are suddenly activated by turning on the Raman laser beams. In Figs. 4a and 4b, we show the time evolution of the fractional spin populations $n_s = \frac{1}{N}\int dk n_s(k)$, as well as the momenta $\langle k\rangle = \sum_s \langle k_s\rangle$ and $\langle \Delta k\rangle = \langle k_2\rangle - \langle k_3\rangle$ (both can be measured by time-of-flight imaging) for $\Phi = 0$, where $n_s(k) = \langle c_s(k)^\dagger c_s(k)\rangle$ and $\langle k_s\rangle = \frac{1}{N}\int k n_s(k) dk$ with N the total atom number. We find that the time evolutions show similar oscillating behaviors for different Φ, but with different frequencies and amplitudes. The difference between the momenta of atoms transferred to state $|2\rangle$ and $|3\rangle$ increase noticeably at early time as a result of the magnetic flux ϕ_0 penetrating the surface of the tube [37], which does not depend on the flux Φ through the tube.

The quench dynamics can also be used to measure the gap closing at phase boundaries. Similar as Ref. [37], we introduce two times τ_2 and τ_3, at which the spin-$|2\rangle$ and spin-$|3\rangle$ populations at $k = 0$ (or $k = \pi$) reach their first maxima, to identify the phase boundary. As we change δ_2 or Ω_{13} across the phase boundary (one gap closes and the dynamics is characterized by a single frequency), τ_2 and τ_3 cross each other. Notice that above discussions only apply to $\Phi = 0, \pi$ where the gap closing occurs. We find that τ_2 and τ_3 are Φ-dependent [55] and would cross each other even when no gap closing occurs for other values of Φ, and the crossing point is generally away from the phase boundaries. Therefore, the measurement of gap closing based on quench dynamics is possible only if Φ can be controlled. As an example, we consider $\Phi = 0$ and plot the time evolution of the spin populations at $k = \pi$ with Ω_{13} around the left phase boundary Ω_{13}^c, as shown in Figs. 4c and 4d.

The ability to lock and control the flux Φ is crucial for the study of both topological properties and quench dynamics. We notice that in the experiment in [37], Φ cannot be controlled and may vary from one experimental realization to another. It is also different for different tubes within one experimental realization. The experimental setup in [37] is shown in Fig. 5a, where three linearly polarized Raman lasers are used to couple three hyperfine spin states of ^{173}Yb atoms (Fig. 5b). It is straightforward to show that the flux $\Phi = 3\varphi_1^\sigma - 2\varphi_2^\pi - \varphi_3^\sigma$, which cannot be controlled since the Raman lasers propagate along different paths, not to mention that their wavelengths are generally not commensurate with the lattice. Moreover, in realistic experiments, arrays of independent fermionic synthetic tubes are realized simultaneously due to the transverse atomic distributions in the y, z directions [36, 37], and the synthetic tubes at different y would have different Φ due to the y-dependent φ_1^σ. For the parameters in Ref. [37], the difference of Φ between neighbor tubes in y direction is about $2\pi \times 0.58$.

Due to the randomness of Φ, the dynamics in experiment [37] (averaged over enough samplings) should correspond to results averaged over Φ. In Fig. 4, we plot the corresponding dynamics averaged over Φ, which show damped oscillating behaviors (with significant long-time damping), as observed in the experiment. The above results still hold for atoms in a weak harmonic trap, as confirmed by the experiment [37] as well as our numerical simulations [55]. In Fig. 4, we fix $\Omega = 6.15J$ and $\delta_2 = -0.2\Omega$. The initial temperature is set to be $T/T_F = 0.3$, with initial Fermi temperature T_F given by the difference between the Fermi energy E_F and the initial band minimum $-2J$ (i.e., $T_F = E_F + 2J$) [37]. We use $T_F = 2J$ to get the similar initial filling and Fermi distribution as those in the experiment (the results are insensitive to T_F). Moreover, τ_2 and τ_3, which cross each other at different values of Ω_{13} for different Φ with averaged value $\bar{\Omega}_{13} = \Omega \neq \Omega_{13}^c$, may not be suitable to identify the phase boundaries (i.e., gap closings) for a random flux Φ. Other experimental imperfections such as spin-selective imaging error may also affect the measured spin dynamics, and the final observed phase boundary in [37] is smaller than Ω and Ω_{13}^c.

Conclusion.— In summary, we propose a simple scheme to realize a controllable flux Φ through the synthetic Hall tube that can be tuned at will, and study the effects of the flux Φ on the system topology and dynamics. The quench dynamics averaged over the random flux may better explain previous experimental results where the flux is not locked. Our results provide a new platform for studying topological physics in a tube geometry with tunable flux and may be generalized with other synthetic degrees of freedom, such as momentum states [56–58] and lattice orbitals [59, 60].

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Supplementary Materials

Quench dynamics with different values of Φ. As we discussed in the main text, even when no gap closing occurs for Φ away from 0 and π, τ_2 and τ_3 would cross each other as we increase Ω_{13}. For different Φ, the crossing points are different and distributed around the averaged value $\Omega_{13}^c = \Omega$. In Fig. S1, we plot the time evolution of the spin populations at $k = \pi$ for different values of Φ.

Quench dynamics with a harmonic trap. In realistic experiments, a harmonic trap is usually applied to confine the atoms. Here we consider its effects on the quench dynamics studied in the main text. From the experimental parameters [1] with a harmonic trap frequency $\omega_x = 2\pi \times 57$ Hz and $J = 2\pi \times 264$ Hz, we obtain the harmonic trap as $V_{\rm trap} = \frac{1}{2} v_{\rm T}\, j^2$ with the trap strength

$v_{\mathrm{T}} \simeq 0.0158J$. In Fig. S2, we plot the time evolution of spin populations with all other parameters the same as in Fig. 3a in the main text. We see that the quench dynamics are hardly affected by the harmonic trap.

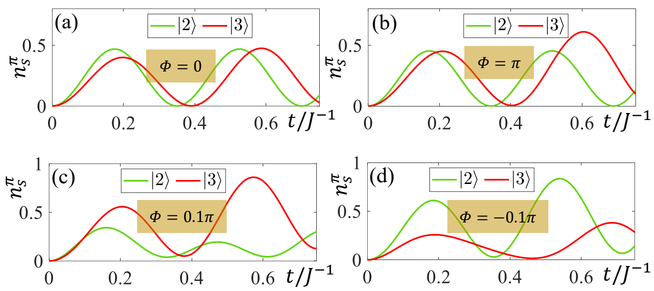

FIG. S1: (a)-(d) Time evolution of the spin populations at $k = \pi$ for different values of Φ. All other parameters are the same as Fig. 3c in the main text.

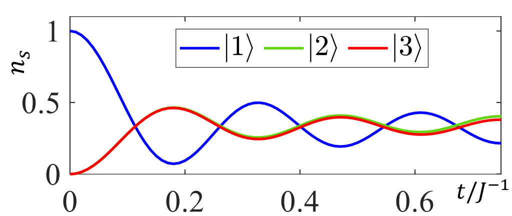

FIG. S2: Quench dynamics in the presence of a harmonic trap $V_{\mathrm{trap}} = \frac{1}{2} v_{\mathrm{T}} j^2$ with trap strength $v_{\mathrm{T}} \simeq 0.0158J$. All other parameters are the same as in Fig. 3a in the main text.

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